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Frege’s Full Comprehension Scheme in Modern Mathematics

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Abstract: Frege’s full comprehension scheme, formulated in *Grundgesetze der Arithmetik*, was central to his logicist project. In its unrestricted form, however, it led directly to Russell’s paradox and was therefore rejected as a foundation for mathematics. This chapter revisits Frege’s comprehension principle from the perspective of modern set theory. Drawing on historical analysis and contemporary mathematical practice, I argue that although full comprehension cannot be sustained as a formal axiom, an informal version continues to guide set-theoretic reasoning. By examining the blurred boundary between sets and proper classes in current mathematical work, I suggest that Frege’s original idea survives as an informal working principle, applied competently by practitioners without generating contradictions.

1. Introduction

Frege’s comprehension scheme was conceived as a cornerstone of his project to ground arithmetic in logic. It asserts that for any property, there exists a corresponding extension—a set or concept-extension—consisting of all objects that satisfy that property. Simple in formulation yet powerful in scope, this principle encapsulates the logicist ambition to construct mathematics from general laws of thought.

The discovery of Russell’s paradox, however, exposed a fatal flaw. Unrestricted comprehension yields contradictions: the class of all classes that do not contain themselves both must and must not contain itself. Confronted with this paradox, Frege acknowledged that his system was inconsistent. In the aftermath, restricted forms of comprehension—most notably the separation scheme of Zermelo–Fraenkel set theory (ZFC)—became central to modern foundational frameworks.

This chapter reconsiders Frege’s conception of comprehension in light of contemporary mathematical practice. I am not a Frege scholar; rather, my professional expertise lies in current set-theoretic practice [9, 10, 11]. It is from this perspective that I approach Frege’s original ideas. I argue that while the full comprehension scheme cannot be reinstated as a formal axiom, it retains substantial conceptual significance. By analysing how set theorists informally employ comprehension when reasoning about sets and models, I suggest that Frege’s original vision persists—not as a formal principle, but as a working principle embedded in the practice of set theory.

Section 2 reviews the historical origins of Frege’s conception of sets and concept-extensions, Russell’s communication of the paradox, Frege’s response, and the subsequent development of strategies aimed at preserving as much comprehension as possible while avoiding inconsistency. One such strategy was incorporated into modern mathematics and is examined in Section 3. There we consider axiomatic set theory and its methods for avoiding Russell’s paradox. I argue, however, that axiomatic frameworks do not fully capture contemporary set-theoretic practice. In particular, the set–class distinction underlying the axiomatic resolution of the paradox proves to be blurred in practice. As a more faithful description of how comprehension operates in modern set theory, I propose an *Informal Comprehension Principle*. Section 4 briefly summarises the chapter.

2. Frege's Full Comprehension Scheme and Russell's Paradox

In Gottlob Frege's account of mathematics, a crucial distinction is drawn between a concept and its extension, *Begriff* and *Umfang* in German. The extension of a concept consists of all objects that fall under that concept. For example, the extension of the concept "red" contains all red objects. Concept-extensions thus constitute an early form of what later came to be called sets. Within this framework, Frege assumed that every concept has an extension—an assumption that, together with other unresolved issues, contributed to one of the most significant crises in the foundations of mathematics, the *Grundlagenkrise*.

The focus of this chapter is Frege's comprehension principle and the question of its survival in contemporary mathematics. This principle is formalised in Basic Law V of *Grundgesetze der Arithmetik* [4, 5]. Basic Law V states that the extension of a concept F is identical to the extension of a concept G if and only if F and G apply to exactly the same objects. In modern set-theoretic notation, this can be expressed as

<u>Basic Law V.</u> $F = G \leftrightarrow \forall x(x \in F \leftrightarrow x \in G)$.

It is important to observe that Basic Law V can be divided into two distinct components:

$$F = G \leftarrow \forall x(x \in F \leftrightarrow x \in G) \qquad \text{(Va)}$$

$$F = G \rightarrow \forall x(x \in F \leftrightarrow x \in G) \qquad \text{(Vb)}$$

Bertrand Russell famously observed that Frege's assumptions give rise to a contradiction. He communicated this result to Frege in a letter written in German. The original text of this letter is reproduced in the appendix. For present purposes, however, I quote from the English translation provided by van Heijenoort [7, pp. 124–125]:

> Friday's Hill, Haslemere, 16 June 1902
>
> Dear colleague,
> For a year and a half I have been acquainted with your *Grundgesetze der Arithmetik*, but it is only now that I have been able to find the time for the thorough study I intended to make of your work. I find myself in complete agreement with you in all essentials, particularly when you reject any psychological element [Moment] in logic and when you place a high value upon an ideography [Begriffsschrift] for the foundations of mathematics and of formal logic, which, incidentally, can hardly be distinguished. With regard to many particular questions, I find in your work discussions, distinctions, and definitions that one seeks in vain in the works of other logicians. Especially so far as function is concerned (§9 of your *Begriffsschrift*), I have been led on my own views that are the same even in the details. There is just one point where I have encountered a difficulty. You state (p. 17) that a function, too, can act as the indeterminate element. This I formerly believed, but now this view seems doubtful to me because of the following contradiction. Let w be the predicate: to be a predicate that cannot be predicated of itself. Can w be predicated of itself? From each answer its opposite follows. Therefore we must conclude that w is not a predicate. Likewise there is no class (as a totality) of those classes which, each taken as a totality, do not belong to themselves. From this I conclude that under certain circumstances a definable collection [Menge] does not form a totality.

> I am on the point of finishing a book on the principles of mathematics and in it I should like to discuss your work very thoroughly [14]. I already have your books or shall buy them soon, but I would be very grateful to you if you could send me reprints of your articles in various periodicals. In case this should be impossible, however, I will obtain them from a library.
> The exact treatment of logic in fundamental questions, where symbols fail, has remained very much behind; in your works I find the best I know of our time, and therefore I have permitted myself to express my deep respect to you. It is very regrettable that you have not come to publish the second volume of your Grundgesetze; I hope that this will still be done.
>
> Very respectfully yours,
> Bertrand Russell

Although Russell had uncovered a serious flaw in Frege's system, the letter makes clear the depth of his admiration for Frege's work.

Frege typically formulated his ideas in terms of what he called *courses of values* of functions. While this terminology did not endure, Frege employed it in his reply to Russell, written six days later. I quote here from the English translation provided by van Heijenoort [7, pp. 127–128]:

> *Jena, 22 June 1902*
>
> *Dear colleague,*
> *Many thanks for your interesting letter of 16 June. I am pleased that you agree with me on many points and that you intend to discuss my work thoroughly. In response to your request I am sending you the following publications:*

1. *"Kritische Beleuchtung",*
2. *"Über die Begriffsschrift des Herrn Peano",*
3. *„Über Begriff und Gegenstand",*
4. *„Über Sinn und Bedeutung",*
5. *„Über formale Theorien der Arithmetik".*

> *[...]*
> *Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic. It seems, then, that transforming the generalization of an equality into an equality of courses-of-values (§9 of my Grundgesetze) is not always permitted, that my Rule V (§20, p. 36) is false, and that my explanations in §31 are not sufficient to ensure that my combinations of signs have a meaning in all cases. I must reflect further on the matter. It is all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish. Yet, I should think, it must be possible to set up conditions for the transformation of the generalization of an equality into an equality of courses-of-values such that the essentials of my proofs remain intact. In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance.*
> *[...]*
> *The second volume of my Grundgesetze is to appear shortly. I shall no doubt have to add an appendix in which your discovery is taken into account. If only I already had the right point of view for that!*

Very respectfully yours,
G. Frege

Russell and Frege continued to exchange several letters in the period that followed, and it quickly became clear that Frege had to respond to the problem. He chose to do so by adding an appendix to the second volume of *Grundgesetze der Arithmetik*. In this appendix, Frege explicitly acknowledged the inconsistency of his system, beginning with the following remark:

> *Einem wissenschaftlichen Schriftsteller kann kaum etwas Unerwünschteres begegnen, als dass ihm nach Vollendung einer Arbeit eine der Grundlagen seines Baues erschüttert wird. In diese Lage wurde ich durch einen Brief des Herrn Bertrand Russell versetzt, als der Druck dieses Bandes sich seinem Ende näherte. Es handelt sich um mein Grundgesetz (V).* [5, p. 253][1]

Frege first introduces the notion of concepts that apply to themselves:

> *Von der Klasse der Menschen wird niemand behaupten wollen, dass sie ein Mensch sei. Wir haben hier eine Klasse, die sich selbst nicht angehört. [...] Fassen wir nun den Begriff ins Auge* Klasse, die sich selbst nicht angehört*!* [5, p. 253f][2]

He then explains the contradiction discovered by Russell, first by presenting an informal derivation and subsequently by providing a formal deduction within his *Begriffsschrift*. Having recognised the severity of the problem, Frege proceeds to consider possible responses and outlines several potential strategies.

One option, Frege suggests, would be to abandon the law of the excluded middle for classes. He quickly dismisses this possibility. Another option would be to allow that there are concepts that lack an extension; however, Frege also rejects this alternative, continuing to assume that every concept has an extension [5, pp. 254–255]. A third possibility is to allow that two concepts may be identical even though not exactly the same objects do fall under both. This amounts to rejecting (Vb), the left-to-right (→) direction of Basic Law V (see above). It is ultimately this strategy that Frege adopts:

> *Der Fehler kann allein in unserm Gesetze (Vb) liegen, das also falsch sein muss. [...] Mit (Vb) ist auch (V) selbst gefallen, nicht aber (Va).* [5, p. 257][3]

From this, he draws the following conclusion:

> *so kommt der Fall vor, dass Begriffe denselben Umfang haben, obwohl nicht alle Gegenstände, die unter den einen dieser Begriffe fallen, auch unter den andern fallen.* [5, p. 260][4]

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[1] Translated by M. Furth [6, p. 127]: "*Hardly anything more unwelcome can befall a scientific writer than that one oft he foundations of his edifice be shaken after the work is finished. I have been placed in this position by a letter of Mr. Bertrand Russell just as the printing of this [second] volume was nearing completion. It is a matter of my Basic Law (V).*"
[2] Translated by M. Furth [6, p. 128]: "*No one will want to assert of the class of men that it is a man. Here we have a class that does not belong to itself. ... Now let us fix our attention upon the concept* class that does not belong to itself."
[3] Translated by M. Furth [6, p. 132]: "*The error can be only in our Law (Vb), which must therefore be false. ... Along with (Vb), (V) itself has collapsed, but not (Va).*"
[4] Translated by M. Furth [6, p. 137]: "*then the case arises of concepts' having the same extension although not all objects falling under one also fall under the other.*"

Here is how rejecting (Vb) blocks the contradiction. Let F be the concept of all concepts that do not apply to themselves, that is, $F = \{x \mid x \notin x\}$. This assumption is placed on the left-hand side of Basic Law V, with G identified as $\{x \mid x \notin x\}$. By rejecting (Vb), however, we can no longer infer from this assumption that the right-hand side of the law also holds. In particular, we cannot derive $\forall x(x \in F \leftrightarrow x \in \{x \mid x \notin x\})$. As a result, neither of the usual assumptions leads to a contradiction:

$$F \in F \ \rightarrow F \notin \{x : x \notin x\} \ \nrightarrow \ F \notin F.$$

$$F \notin F \rightarrow F \in \{x : x \notin x\} \ \nrightarrow \ F \in F.$$

In both cases, the first implication is secured by the definition of F, while the second implication is blocked by the rejection of (Vb). Consequently, F and $\{x \mid x \notin x\}$, although equal by definition, need not contain exactly the same elements.

While this strategy does succeed in avoiding Russell's contradiction, it does so at the cost of a highly counterintuitive conception of equality. For this reason, it was not pursued further in the subsequent development of the foundations of mathematics.

Instead, a number of alternative strategies were developed to address the problem. Most prominently, Bertrand Russell, together with Alfred North Whitehead, introduced a theory of types in *Principia Mathematica* [17]. Roughly speaking, if extensions, or sets, are arranged hierarchically into different types, then an extension cannot be a member of itself but can only contain elements of a lower type. Under such a regime, defining $F = \{x \mid x \notin x\}$ does not give rise to a meaningful question as to whether $F \in F$.

Versions of type theory have survived in various areas of mathematics. Notably, homotopy type theory provides an alternative foundation for mathematics, known as the univalent foundations, which were developed and vigorously promoted by Fields Medalist Vladimir Voevodsky. In addition, philosophical logic has produced further typed frameworks, such as typed theories of truth [15].

Another influential approach was developed by George Boolos and led to the notion of plural logic. The central idea is that a plurality, unlike a set, is nothing over and above the objects that compose it. This contrasts sharply with sets, since, for example, 1 and $\{1\}$ are distinct objects. Forming a set introduces a new object, whereas considering objects as forming a plurality does not. In his original paper, Boolos illustrated this idea—perhaps partly for amusement—by appealing to a bowl of Cheerios:

> It is haywire to think that when you have some Cheerios, you are eating a *set*—what you're doing is: eating THE CHEERIOS. [2, emphases original, p. 448]

Plural logic supports a slightly modified form of comprehension:

$$\exists u \varphi(u) \rightarrow \exists xx \forall u\big(u \prec xx \leftrightarrow \varphi(u)\big)$$

This principle states that if there exists an object satisfying the property φ, then there exists a plurality consisting of exactly those objects that satisfy φ. Plural logic remains an active area of research in the philosophy of mathematics. Nevertheless, it has not been integrated into contemporary set-theoretic practice.

A third approach worth mentioning seeks to resolve Russell's paradox by adopting a paraconsistent logic. There exist several formal systems of paraconsistent set theory that retain a full comprehension principle, developed, for example, by Newton da Costa [3], Graham Priest [13], and Zach Weber [16]. The central idea of paraconsistent systems is to block the principle of explosion, which in classical logic allows any statement to be inferred from a contradiction. In classical logic, from a contradiction A and $\neg A$, any conclusion B follows:

- Premise 1: A
- Premise 2: $\neg A$
- Inference 1; from premise 2 by ∨-Introduction: $\neg A \lor B$
- Conclusion; from premise 1 and inference 1 by Modus Ponens: B

In paraconsistent set theories, by contrast, contradictions—such as those generated by Russell's paradox—are contained and do not license arbitrary inferences. A common motivation for paraconsistent logic is that it more closely reflects actual human reasoning: when people encounter contradictions, they do not thereby accept every statement as true. From this perspective, paraconsistent logic is taken to model real reasoning practices more faithfully. While this remains a vibrant research area in logic and philosophy, it has not been adopted within mainstream mathematical practice.

Overall, Frege's failure to provide a consistent logicist foundation for mathematics marks a decisive turning point in the history of logic and set theory. Nevertheless, his comprehension principle continued to shape subsequent developments, most notably through its transformation in Zermelo's 1908 axiomatization of set theory [18]. There, full comprehension was replaced by *Aussonderung*, or separation—a restricted principle applying only to subsets of already given sets. This strategy, and its role in modern set theory, is the focus of the remainder of this chapter.

3. Comprehension and Modern Set Theory

Modern set theory inherits the ambition underlying Frege's comprehension principle, but tempers it with the restrictions required for consistency. The formal standard theory of sets is Zermelo-Fraenkel set theory with the axiom of choice, abbreviated ZFC. In ZFC set theory, the unrestricted formation of a set of all objects satisfying a given property is replaced by the more cautious separation principle. According to this principle, given any set and any property, there exists a subset consisting of exactly those elements of the given set that satisfy the property (see Figure 1).

Separation scheme (*Aussonderung*). For every first-order formula $\varphi(x)$:
$\forall z \exists y \forall x (x \in y \leftrightarrow \varphi(x) \land x \in z)$.

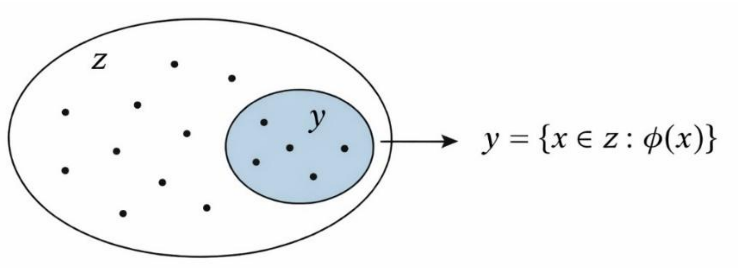

Figure 1: Separation scheme (*Aussonderung*)

The separation scheme avoids paradoxes by grounding comprehension within already existing sets.

At the same time, modern set theory draws a formal distinction between sets and proper classes, while retaining an unrestricted form of comprehension for classes:

Class comprehension. For every first-order formula $\varphi(x)$, there is a class C containing exactly those sets that satisfy $\varphi(x)$. Formally, $\forall x(x \in C \leftrightarrow \varphi(x))$.

The crucial distinction between sets and proper classes is that proper classes cannot be elements of anything. Proper classes are regarded as too large to be sets. Consequently, if a class C defined by class comprehension is a proper class, the question whether $C \in C$ does not arise. In this way, Russell's paradox is avoided. Compared to Frege's framework, modern set theory assumes that every concept has a class extension, though not necessarily a set extension.

Class comprehension is not an axiom of ZFC. However, commonly used class-theoretic axiomatisations, such as NBG or MK, include formal axioms governing proper classes, including versions of class comprehension. Von Neumann–Bernays–Gödel (NBG) set theory is an axiomatic system that extends ZFC by introducing classes and distinguishing between sets and proper classes. Morse–Kelley (MK) class theory is a stronger axiomatic foundation that treats classes as fundamental objects; in MK, sets are defined as those classes that can be members of other classes. This makes MK more powerful than standard set theories such as ZFC or NBG. By contrast, ZFC itself includes axioms only about sets.

Among the most commonly used and most important proper classes in set theory are V, Ord, and L. The class V denotes the universe of all sets and is defined by $V = \{x \mid x = x\}$. Since every set is identical to itself, every set belongs to V by definition. The class of ordinal numbers is denoted by Ord and defined as $Ord = \{x \mid x \text{ is an ordinal}\}$. The ordinals, introduced by Georg Cantor, begin with the natural numbers $0,1,2,3, \ldots$, continue in the infinite with $\omega, \omega + 1, \omega + 2, \ldots$, and extend far beyond. There are so many ordinals that they cannot form a set; instead, they constitute a proper class. A third widely used proper class in modern set theory is the constructible universe L. This is a class-sized model of ZFC, introduced by Kurt Gödel to establish the consistency of the continuum hypothesis relative to ZFC and the consistency of the axiom of choice relative to ZF. This proper class is defined as $L = \{x \mid x \text{ is constructible}\}$. Figure 2 illustrates these proper classes.

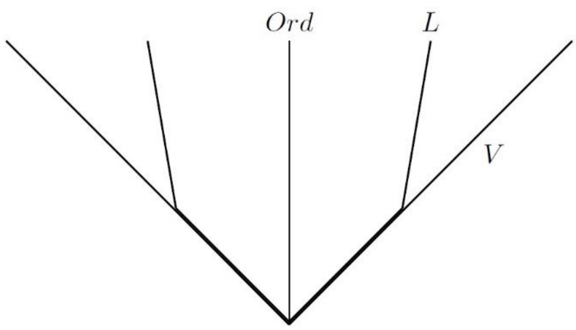

Figure 2: The proper classes, V, Ord, and L.

The quick solution Frege proposed in the appendix to *Grundgesetze*—namely, rejecting one direction of Basic Law V—was abandoned in modern set theory in order to preserve an intuitive notion of equality. Instead, both directions of Basic Law V are retained: two sets are equal if and only if they have exactly the same elements. This principle is known as the extensionality of sets and captures a basic feature of what it means to be a set:

<u>Extensionality.</u> $\forall y \forall z(y = z \leftrightarrow \forall x(x \in y \leftrightarrow x \in z))$.

Extensionality by itself does not generate a contradiction, since modern set theory does not allow the formation of the Russell set. A key difference from Frege's conception of sets as concept-extensions is that, in modern set theory, sets are not understood as the extensions of concepts. Rather, sets are characterised solely by their elements, as extensionality requires, and not by properties used to define them.

Overall, modern set theory rejects the full set comprehension principle. At the formal level, it instead relies on class comprehension, the separation scheme, and the extensionality axiom. However, this formalisation captures only an impoverished version of actual set-theoretic reasoning. Indeed, working set theorists themselves disagree on how natural or faithful this formal framework is as a representation of their everyday mathematical practice.

The set theorist and logician Charles Pinter observes:

> the distinction between classes which are elements [sets] and [proper] classes which are not elements [is] a highly artificial one […]
> the amount of intuitive set theory which [Zermelo's and von Neumann's systems][5] had to sacrifice was considerable. For example, in Zermelo's system, as we have already seen, the intuitive way of making sets—by naming a property of objects and forming the set of all objects which have that property—does not take place at all. [12, p. 20]

Pinter argues that the distinction between sets and proper classes is highly artificial. He also maintains that Frege's comprehension principle is deeply intuitive, and he expresses dissatisfaction with the extent to which it is restricted in formal set theory.

From a different perspective, the philosopher of set theory Luca Incurvati contends that full comprehension is not, in fact, intuitive. Rather, he argues that the apparent appeal of set comprehension arises from a conflation of two distinct concepts:

> [a] possible explanation for the fact that we make the naïve supposition that all conditions determine a set is that we are conflating two concepts, the concept of set and the concept of objectified property. [...]
> Once the two conceptions are clearly distinguished [...], we can explain the appeal of the Naïve Comprehension Schema without having to accept its truth for sets. [8, p. 76]

This once again points to Frege's conception of sets as concept-extensions, as opposed to sets defined solely by their elements. These remarks indicate that the question of comprehension remains unsettled within modern set theory.

One might assume that the formal principles of modern set theory outlined above provide an accurate description of how set theory is actually practiced. On this view, the accepted formal principles—class comprehension, *Aussonderung*, and extensionality—would faithfully capture the working methods of set theorists. This perspective naturally encourages the following assumptions:

1. There are proper classes for which unrestricted comprehension holds.

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[5] Von Neumann's system is a class theory, formalising, among others, principles such as class comprehension, see NBG above.

2. There are sets for which only restricted comprehension, in the form of *Aussonderung*, holds.
3. Proper classes and sets are clearly distinguished kinds of objects.

I will argue, however, that assumption (3) must be rejected when confronted with actual set-theoretic practice. The crucial observation is that, in everyday research, set theorists routinely move between treating certain collections as proper classes and as sets, simply by shifting the surrounding framework or perspective.

To illustrate how this works, I will present two examples.

Example 1: Adding large cardinals.

If we consider the set-theoretic universe V and then assume the existence of an inaccessible cardinal κ (or another sufficiently large cardinal), the proper class V can be treated as a set—namely, as the initial segment V_κ. Figure 3 illustrates this standard working practice.

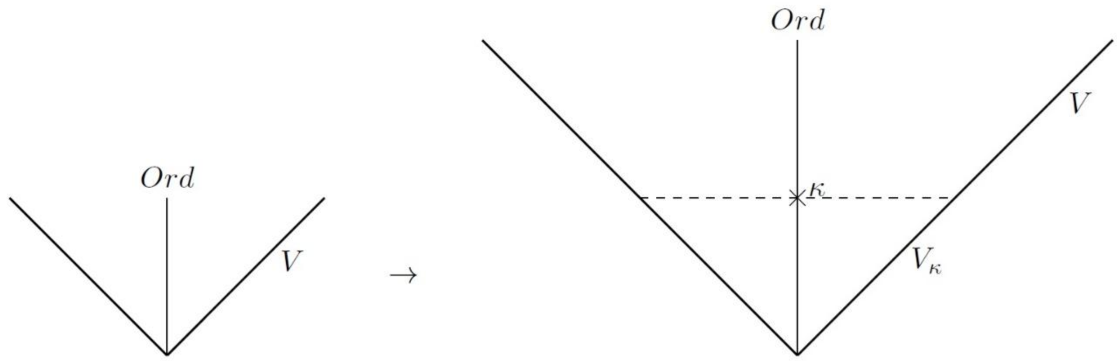

Figure 3: Adding an inaccessible κ – from the proper class V to the set V_κ.

In the extended theory ZFC +“there exists an inaccessible cardinal κ,” we can prove that what previously functioned as Vcan now be regarded as a set:

$$\text{ZFC} + \text{“there exists an inaccessible } \kappa\text{”} \ \vdash \ \exists x(x = V_\kappa = V^{V_\kappa}).$$

The previously class-sized universe V is a model of ZFC, just as the set V_κ is a model of ZFC in the extended theory. From the standpoint of the mathematical structure itself, nothing has changed; what changes is our perspective, brought about by the assumption of an inaccessible cardinal.

Set theorists generally regard the existence of inaccessible cardinals—among the smallest large cardinals—as a consistent and legitimate assumption. As such, appealing to inaccessible cardinals is a standard and widely accepted practice in contemporary set-theoretic reasoning.[6]

Example 2: Models of set theory.

More generally, set theorists routinely work with models of set theory [1, 10]. Each such model is a set from an external perspective, yet from within the model it appears as the entire set-theoretic universe.

[6] (Kant 2025c, Sec. 3.2) elaborates on the use of large cardinal axioms in set-theoretic practice for various purposes and explains the underlying consistency assumptions.

Within a set-sized model M of ZFC itself, the classes V, Ord, and L are proper classes rather than sets. Accordingly, from the internal perspective of M, there are no such sets:

$$M \vDash \nexists x\,(x = V),$$

$$M \vDash \nexists x\,(x = \mathrm{Ord}),$$

$$M \vDash \nexists x\,(x = L).$$

By shifting perspective and considering M from the outside, however, we regard M itself as a set, and consequently V^M, Ord^M, and L^M as subsets of M, as illustrated in figure 4.

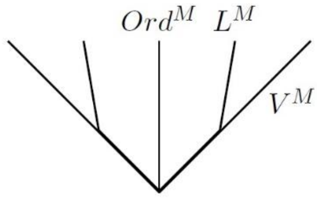

Figure 4: The sets, V^M, Ord^M, and L^M.

From this external standpoint, what were proper classes internally are treated as sets. Indeed, we can prove:

$$\mathrm{ZFC} \vdash \exists x\,(x = V^M),$$

$$\mathrm{ZFC} \vdash \exists x\,(x = \mathrm{Ord}^M),$$

$$\mathrm{ZFC} \vdash \exists x\,(x = L^M).$$

In all three cases, the underlying objects remain the same; what changes is solely our perspective on them.

There are many further concrete examples. One prominent case is the countable model approach to forcing. When the forcing method is applied to a countable model of set theory—the so-called ground model—V, Ord, and L are proper classes from the internal perspective of the ground model, but they appear as sets from an external standpoint. This shift in status is essential to the execution of the forcing method itself.

The underlying reason why such perspective shifts are possible lies in the phenomenon illustrated by Example 2, which is a central feature of modern set-theoretic practice. Because object-level and meta-level considerations are closely intertwined, and because set theorists routinely move between these levels, the same mathematical object may be treated as a set in one context and as a proper class in another, or vice versa. These practices indicate that, in contemporary set theory, the set–class distinction is not fixed once and for all; rather, it is context-dependent, and sets and proper classes are not sharply distinguished kinds of objects in practice.

It is important to emphasise, however, that set theorists are well aware of this indeterminacy and are able to manage it without risking inconsistency. Each shift of perspective can be made precise and, when necessary, fully formalised.

These observations motivate the formulation of an informal comprehension principle, which my analysis suggests accurately reflects contemporary set-theoretic practice.

Informal Comprehension Principle.

For every first-order formula φ, there exists a class C containing exactly those sets that satisfy φ; by class comprehension: $C = \{x \mid \varphi(x)\}$). By passing to an appropriate model of set theory M, the class C can be treated as a set, namely as the corresponding subset C^M. This shift of perspective is standard practice in set-theoretic research.

On this view, comprehension remains operative at an informal level. The restriction from Frege's full comprehension principle to the separation scheme of ZFC does not represent a philosophical retreat, but rather a technical safeguard. The methodological intuition that "every property determines a collection" continues to underwrite set-theoretic reasoning in practice.

4. Conclusion

As is well known, Frege's assumption that every concept has an extension, together with his Basic Law V, lead to contradiction. In response to this failure, modern axiomatic set theory abandoned full comprehension as a formal principle, replacing it with restricted comprehension (*Aussonderung*), class comprehension, and a principled distinction between sets and proper classes. At the level of formal foundations, this distinction is treated as sharp and indispensable for consistency.

This chapter has argued, however, that the formal picture does not fully capture contemporary set-theoretic practice. In actual mathematical work, the distinction between sets and proper classes is fluid and context-dependent. Set theorists routinely treat one and the same mathematical object as a proper class from one perspective and as a set from another, most notably when shifting between internal and external viewpoints on models of set theory. Examples involving large cardinals, models, and forcing show that this perspectival flexibility is not marginal, but a standard and indispensable feature of modern set-theoretic reasoning.

From this observation, I have drawn a modest but significant conclusion. Frege's full comprehension scheme has indeed been abandoned as a formal axiom, but its guiding intuition has not disappeared. Instead, it survives as what I have called an *Informal Comprehension Principle*. In practice, set theorists freely form classes defined by arbitrary first-order properties and, by passing to suitable models, just as freely regard these classes as sets when doing so is methodologically useful and formally harmless. Comprehension thus remains operative at an informal level, carefully managed through shifts of perspective rather than rigidly fixed ontological categories.

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Appendix: Russell's original letter to Frege

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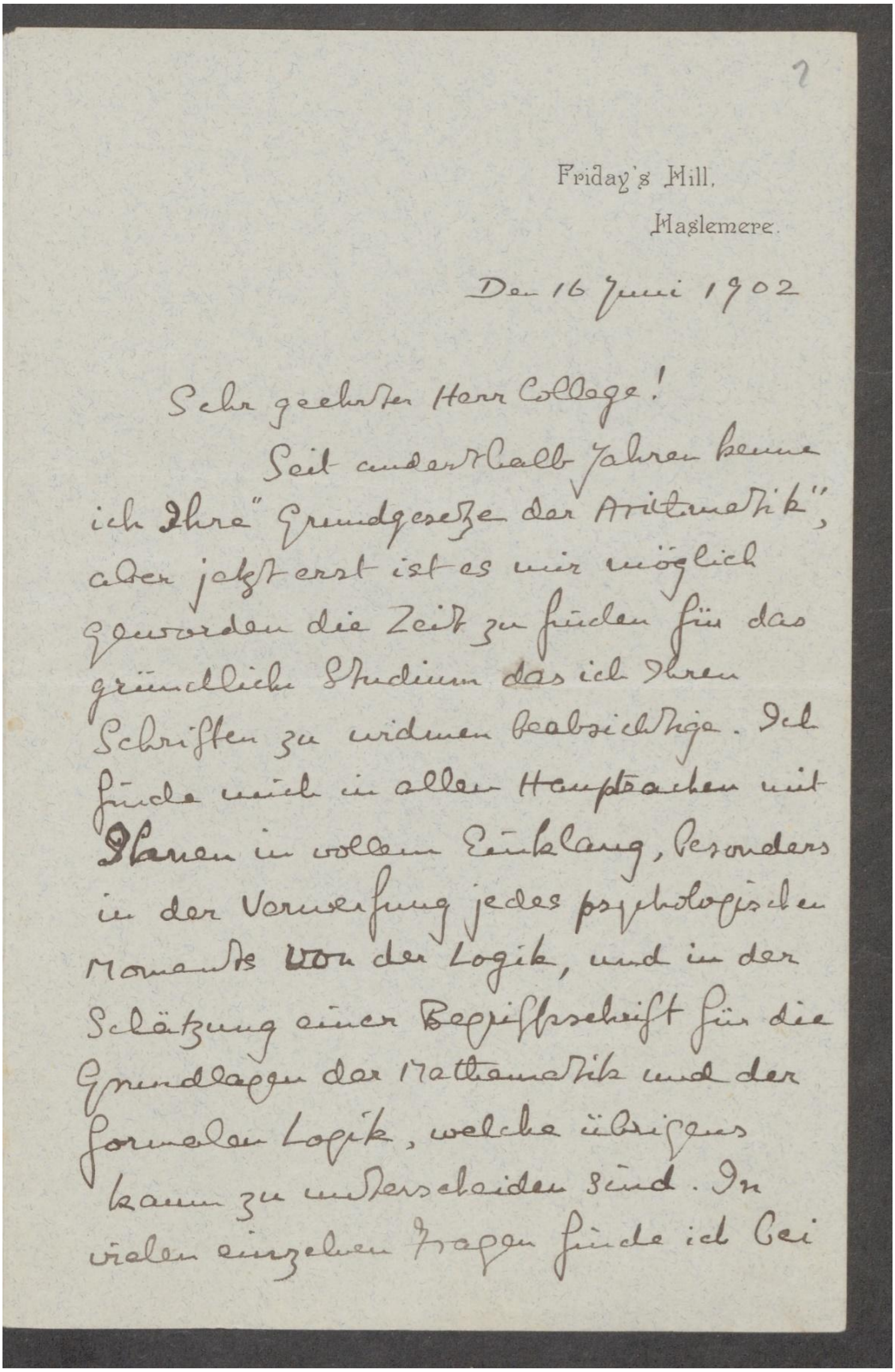

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Friday's Hill,
Haslemere.

Den 16 Juni 1902

Sehr geehrter Herr College!

Seit anderthalb Jahren kenne ich Ihre "Grundgesetze der Arithmetik", aber jetzt erst ist es mir möglich geworden die Zeit zu finden für das gründliche Studium das ich Ihren Schriften zu widmen beabsichtige. Ich finde mich in allen Hauptsachen mit Ihnen in vollem Einklang, besonders in der Verwerfung jedes psychologischen Momentes von der Logik, und in der Schätzung einer Begriffsschrift für die Grundlagen der Mathematik und der formalen Logik, welche übrigens kaum zu unterscheiden sind. In vielen einzelnen Fragen finde ich bei schliessen dass w kein Prädicat ist. Ebenso giebt es keine Klasse (als Ganzes) derjenigen Klassen die als Ganze sich selber nicht angehören. Daraus schliesse ich dass unter gewissen Umständen eine definierbare Menge kein Ganzes bildet.

Ich bin im Begriff ein Buch über die Prinzipien der Mathematik zu vollenden, und ich möchte darin Ihr Werk sehr ausführlich besprechen. Ihre Bücher habe ich schon, oder ich kaufe sie bald; aber ich wäre Ihnen sehr dankbar wenn Sie mir Sonderabdrücke Ihrer Artikel in verschiedenen Zeitschriften schicken könnten. Falls dies aber unmöglich sein sollte, so schaffe ich sie mir

Ihnen Discussionen, Unterscheidungen, und Definitionen, die man vergebens bei anderen Logikern sucht. Besonders über die Funktion (§ 9 Ihrer Begriffsschrift) bin ich bis ins Einzelne selbständig zu denselben Ansichten geführt worden. Nur in einem Punkte ist mir eine Schwierigkeit begegnet. Sie behaupten (S. 17) es könne auch die Funktion das unbestimmte Element bilden. Dies habe ich früher geglaubt, jedoch jetzt scheint mir diese Ansicht zweifelhaft wegen des folgenden Widerspruchs: Sei <u>w</u> das Prädicat, ein Prädicat zu sein welches von sich selbst nicht prädicirt werden kann. Kann man <u>w</u> von sich selbst prädiciren? Aus jeder Antwort folgt das Gegentheil. Deshalb muss man aus einer Bibliothek.

Die exacte Behandlung der Logik, in den Fundamentalfragen, wo die Symbolen versagen, ist sehr zurückgeblieben; bei Ihnen finde ich das Beste, was ich aus unserer Zeit kenne, und deshalb habe ich mir erlaubt, Ihnen mein tiefes Respekt auszudrücken. Es ist sehr zu bedauern, dass Sie nicht dazu gelangt sind, den zweiten Band Ihrer Grundgesetze zu veröffentlichen; hoffentlich wird das doch noch geschehen.

Mit hochachtungsvollem Grusse,

Ihr ergebenster

Bertrand Russell.

Obiger Widerspruch drückt sich in Peano's Begriffsschrift wie folgt aus:

$$w = \overset{\text{cls} \cap}{\hat{x}} \ni (x \sim\varepsilon x) . \supset : w \varepsilon w . = . w \sim\varepsilon w .$$

Ich habe darüber an Peano geschrieben, aber er bleibt mir eine Antwort schuldig.

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