Forcing in Set-Theoretic Practice

Deborah Kant

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This chapter examines the philosophical significance of the forcing method through the lens of contemporary set-theoretic practice. Building on the practice-based approach pioneered by Penelope Maddy, it situates forcing within the broader universe–multiverse debate and critically compares the universe view defended by Maddy with Joel D. Hamkins' multiverse pluralism. Although both accounts appeal to mathematical practice, they draw divergent conclusions, raising a methodological puzzle about how practice is to be interpreted. To address this puzzle, the chapter introduces a scope guideline for practice-based philosophy of set theory, according to which philosophical analyses must respect and explain the full diversity of set-theoretic practices rather than privileging a narrow subset. The guideline is applied in a detailed case study of forcing, drawing on qualitative interview data from professional set theorists representing both absolutist and pluralist orientations. The findings show that forcing is widely valued across the community but integrated into distinct research aims. The chapter argues that forcing is largely philosophically neutral, though not uniformly so, and that its methodological versatility reflects the coexistence of multiple epistemic ideals within set theory. More generally, the analysis highlights how practice-based philosophy must accommodate internal diversity if it is to yield credible philosophical conclusions about mathematics.

1 Introduction

The method of forcing has long stood at the heart of modern set theory. Since its introduction by Paul Cohen in the 1960s, forcing has not only transformed the mathematical landscape—showing, for example, that the Continuum Hypothesis (CH) cannot be decided within ZFC—but has also become a focal point for philosophical reflection. Forcing provides a striking case of how mathematical techniques can shape, and be shaped by, competing conceptions of mathematical reality.

The aim of this chapter is to examine the philosophical significance of forcing through the lens of set-theoretic practice. Recent decades have seen a turn toward practice-based philosophy of mathematics, which seeks to understand mathematical knowledge by analysing what mathematicians actually do, rather than by appealing solely to abstract foundations. Within set theory, this shift has revealed a striking methodological and philosophical diversity. While some researchers—so-called absolutists—seek to extend ZFC with new true axioms and preserve the ideal of a single set-theoretic universe, others—pluralists—embrace the multiplicity of models that forcing makes possible.

Two leading philosophical accounts illustrate this divide. Penelope Maddy's formulation of the universe view interprets set-theoretic practice as supporting the search for a single, standard axiomatic theory of sets. Joel D. Hamkins' multiverse view, by contrast, interprets the very same practices—especially the use of forcing and model construction—as evidence that there are many legitimate set-theoretic worlds. Both appeal to mathematical practice, yet they draw divergent philosophical conclusions. This tension raises an important methodological question: how can the same body of practice yield such different interpretations?

To address this question, the chapter proceeds in three steps. First, I review the philosophical background including Maddy's position, philosophical discussions involving forcing, and Hamkins' position (§2). Second, I propose a scope guideline for interpreting set-theoretic practice, which emphasises the importance of capturing the full diversity of set-theoretic research (§3). Finally, I apply this framework to an empirical case study on the practice of forcing (§4), drawing on qualitative data from interviews with professional set theorists. This empirical perspective helps to illuminate how forcing functions within distinct research traditions and whether its use can be considered philosophically neutral.

In doing so, the chapter aims not to settle the universe–multiverse debate, but to clarify how philosophical conclusions about set theory depend on the interpretive frameworks we bring to its practice. Forcing, as we shall see, exemplifies both the unity and the diversity of contemporary set-theoretic research—and thus serves as a revealing case for understanding how mathematics and philosophy interact.

2 Philosophical Background: Set-Theoretic Practice, Forcing, and the Universe-Multiverse Debate

2.1 Philosophical Paths to Set-Theoretic Practice

The philosophy of mathematics has long been an important field concerned with central questions of ontology and epistemology. In the early 20th century—marked by the foundational crisis in mathematics—debates about the nature of mathematical truth were especially lively. Significant progress arose from both formal and philosophical developments, culminating in the classical programs of formalism, logicism, and intuitionism. Over time, however, some scholars argued that philosophical discussions had become insufficiently connected to actual mathematical practice. For example, identifying mathematical proof with formal derivation fails to capture the way mathematicians really work, since they rarely spell out their arguments in full formal detail. The central critique was that the philosophy of mathematics had come to rely on overly idealised concepts.

A number of influential figures advanced a new direction—now known as the philosophy of mathematical practice. Imre Lakatos [1976] emphasised the historical development of mathematics, showing how definitions and proofs co-evolve. Paolo Mancosu [2008] edited a seminal volume that broadened the field by introducing topics such as visualisation, explanation, and value judgements in mathematics. Maddy [1997] focused on set-theoretic practice and argued for a direct connection between the concerns of working set theorists and the justification of axioms. Since then, the philosophy of mathematical practice has developed into a well-established subfield, represented institutionally by the Association for the Philosophy of Mathematical Practice.[1]

Meanwhile, a parallel development in the 20th century made set theory an especially significant discipline for philosophers of mathematics. Introduced by Georg Cantor in the late 19th century, set theory was accompanied from its inception by deep philosophical puzzles, including Russell's Paradox and questions about large infinities that were entirely new to both mathematicians and philosophers. Despite these challenges, the advantages of the new theory quickly secured its place as an important area of research. Cantor's work on point sets illustrated its value for studying the real numbers, and set theory soon proved capable of serving as a foundational framework for all of mathematics. Mathematical objects could be represented uniformly as sets, and the emerging axiomatization of set theory provided a general account of mathematical proof. This foundational role, together with its close ties to logic, made set theory particularly attractive to philosophers interested in the nature of mathematical truth and justification.[2]

Today, both the philosophy of mathematical practice and the philosophy of set theory are well-established subfields within the broader philosophy of mathematics. In what follows, I focus on the philosophical implications of one of set theory's most distinctive methods—forcing—and examine how different interpretations of this practice shape contrasting philosophical conclusions.

Let me say a few words about the content of contemporary set-theoretic research and what we look at to study its practices. Set theory is a mathematical subdiscipline situated at the intersection of mathematical logic and foundational studies, yet in many respects it operates like any other field of mathematics. Set theorists typically work within the axiomatic framework of ZFC and its extensions by large cardinal axioms, forcing axioms, determinacy principles, and other mathematically motivated statements. A distinctive feature of the discipline is that set theory provides its own meta-semantic environment: models of axiomatic theories are themselves sets, and set theorists routinely study these

[1]More overview information on the field can be found in Sec. 2 in D'Alessandro's chapter, the recently published SEP entry [De Toffoli and Mancosu, 2026], [Carter, 2025], and [Rittberg, 2019]. In this book, the chapters by Arana, Avigad, Baker, Bangu, Chemla, Colyvan, De Toffoli and Tanswell, Maddy, McLarty, Paseau and Pregel, Reck, and Schlimm and Arana [REFERENCE] address contemporary questions in the philosophy of mathematical practice.

[2]The chapters by Hamkins and Studd [REFERENCES] in this book develop philosophical questions concerning set theory.

models as mathematical objects. Much contemporary set theory therefore centres on the construction, comparison, and analysis of models of ZFC and stronger theories. Questions of consistency and consistency strength play a central role. A typical research question asks whether a given statement *can* be true—whether there is a model in which it holds—or whether it must fail in all models satisfying certain axioms. For example, CH cannot hold in models of the Proper Forcing Axiom (PFA), since PFA implies its negation $\neg$CH. In the area of cardinal invariants of the reals, any two cardinal invariants *can* often be separated by forcing: set theorists can construct a—usually technologically highly complex—forcing model in which the two cardinal invariants have different sizes.[3] In a prominent case, however, set theorists proved that two cardinal invariants can *not* be separated.[4] Other research programmes investigate canonical inner models for large cardinals, the interaction between forcing and large cardinals, the consequences of forcing axioms, and a wide range of related constructions.

Understanding set-theoretic practice requires attention not only to technical methods but also to the broader range of activities in which set theorists engage. These include developing proof techniques, formulating conjectures, searching for proofs or counterexamples, and identifying promising research directions. Such activities are guided by a network of mathematical convictions—about which hypotheses are true, which axioms are natural, or which principles merit further study—as well as epistemological views concerning what counts as justification in set theory. They are also shaped by explicitly philosophical questions about the subject matter of set theory, the aims of foundational research, and the relation between set-theoretic constructions and the mathematical or physical world.

Mathematical practice is expressed through diverse forms of communication that philosophers can study: research articles, preprints, textbooks, lectures, seminar talks, and informal conversations among colleagues. Broader perspectives—such as sociological analyses of the development of the research community, historical studies of concept formation and standards of rigour, and cognitive investigations of how mathematicians understand abstract notions like sets, numbers, or infinity—further enrich the picture. Through these ways, philosophers of mathematical practice can study a complex and dynamic practice of considerable interest.[5]

The philosophical study of set-theoretic practice is most closely associated with Penelope Maddy, who developed the first and still most comprehensive account. Other scholars explicitly focusing on set-theoretic practice include Colin Jakob Rittberg [2015, 2016, 2024], who has analysed the role of metaphysical convictions in shaping mathematical research programmes and criticised Maddy's account for being too narrow. Carolin Antos has examined explanation in set-theoretic proofs—a paradigmatic topic in the philosophy of mathematical practice with an extensive literature—and argued for a metaphysically weaker view of set-theoretic models than the one defended by Hamkins [Antos, 2022,

[3]A cardinal invariant is the size of a specific uncountable set of reals or subsets of reals, i.e. a cardinal invariant has at least size $\aleph_1$, but can be much larger.

[4]The cardinal invariants $\mathfrak{p}$ and $\mathfrak{t}$ are known to coincide [Malliaris and Shelah, 2013].

[5]For a reflection on how the philosophy of mathematical practice should proceed, see [D'Alessandro, 2025].

Antos and Colyvan, 2024]. There is additional work that draws on arguments or case studies from set-theoretic practice, but here the focus is on approaches whose *primary* argumentative force derives from considerations of such practice. To prepare the ground for a more detailed analysis of how forcing is interpreted within set-theoretic practice, we begin by examining Maddy's account.

2.2 Maddy's Axiomatic Universe View and the Role of Practice

A central account in the philosophy of set theory is provided by Maddy [1988a,b, 1997, 2011, 2017, 2024, 2026], who has worked for four decades on the subject and emphasises the special significance of set-theoretic practice. Summarising its key features, her account integrates

* a philosophical methodology: Second Philosophy,
* an epistemology of set theory: which understands proper set-theoretic methods in terms of means–ends relations,
* an ontology of set theory: Arealism or Thin Realism,
* a basis for mathematical objectivity: mathematical depth, and
* a defence of an axiomatic universe view appealing to set theory's foundational role.

The substantive fundament of Maddy's account is its epistemology, in which set-theoretic practices directly suggest philosophical conclusions. Her approach illustrates what a naturalist philosophy grounded in mathematical practice can achieve: it can yield a comprehensive philosophical theory of mathematics. Maddy has been one of the pioneers in incorporating considerations of mathematical practice into a broader philosophical framework, and her analysis has become so influential that contemporary accounts of set-theoretic practice must situate themselves in relation to hers.

Let us begin with Maddy's philosophical methodology: Second Philosophy, which is a specific form of naturalism. Maddy characterises this methodology through a persona—the Second Philosopher—who investigates the world using the tools of ordinary scientific inquiry, and a crucial aspect of this approach is the ability to subject her own observations and ideas to critical scrutiny, discarding those that fail to cohere. Part of her endeavour is then to understand how set theory operates.

To build up set-theoretic epistemology in terms of means–ends relations, Maddy analysed a large variety of specific effective means and desired mathematical goals. An important method—especially in set theory—is the adoption of axioms. Maddy asks: on what grounds are axioms adopted? She examines this question through several case studies involving the ZFC axioms, large cardinal axioms, and determinacy principles [Maddy, 1988a,b, 2011].

As an example, consider Projective Determinacy (PD). The theory ZFC can neither prove nor refute PD, but one could adopt it in addition to ZFC. The statement PD says that all projective sets of reals are determined, i.e. for a two-player game on any projective set, one of the players has a winning strategy. Maddy argues that adopting PD is an effective means toward several mathematical goals, one of which is to obtain a generically absolute theory for $H(\omega_1)$.[6] Maddy extracts such goals directly from set-theoretic practice by quoting set theorists, and she employs the following procedure [Maddy, 1997, 194]:

1. Extract consensus-based mathematical goals from the practice.
2. Extract potential means to achieve these goals from the practice, including potential means–ends relations.
3. Examine whether the potential means–ends relation is reasonable.
   * If yes, classify the method as proper set-theoretic methodology.
   * If not, discard the consideration as *not* belonging to proper set-theoretic methodology.

Following this procedure, the Second Philosopher comes to understand how set theory operates.[7]

The consensus mentioned in step 1 is essential. But this raises a worry. If consensus in the set-theoretic community is central to Maddy's philosophical conclusions, can we trust that such consensus cannot go astray? Is it not possible for a community to adopt false or misguided beliefs? Maddy acknowledges this worry in what she herself calls a nihilistic footnote [Maddy, 1997, 198, fn. 9]. In her later work, she proposes a response. Maddy [2011] introduces an objective standard for mathematics: *mathematical depth*. With this criterion in place, community consensus can indeed be mistaken. If set theorists pursue a goal that fails to lead to deep mathematics, then that goal does not qualify as part of proper set-theoretic methodology. On this view, mathematical practice functions as a channel through which knowledgeable practitioners convey what is mathematically deep, even if this communication can occasionally misfire. This provides an epistemology of objective mathematics.

With respect to the central debate in the philosophy of set theory between the *universe view* and *set-theoretic pluralism*, Maddy takes a definite stance. Her axiomatic universe view holds that there is a single, unified theory of sets in which open set-theoretic questions are answered. Set-theoretic pluralism, by contrast, denies this: ZFC permits the study of many axiomatic extensions of ZFC, and no single of these theory extensions is more "correct" than the others. Maddy notes that one of the aims of set theory is to provide a foundation for mathematics, and argues in favour of the universe view. For a unified theory is an effective means toward this goal because it yields definite answers to mathematical questions. This is Maddy's principle UNIFY [Maddy, 1997, 208–212]. On

[6] $H(\kappa)$ denotes the set of all sets that are hereditarily of cardinality less than κ. The theory of $H(\omega_0)$ is the theory of finite sets and ZFC provides all necessary axioms for it. However, the theory of $H(\omega_1)$ is underdetermined by ZFC. Adopting PD would make this theory immune to forcing (generically absolute), meaning that one cannot change the theory by common forcing methods. For a comprehensive list of the other reasons in favour of PD, see [Maddy, 2011, 47–51].

[7] More details about Maddy's methodology are included in a previous analysis [Antos and Kant, 2024].

this view, set-theoretic questions—such as the continuum problem—can receive definite answers by adding extrinsically justified axioms to ZFC:

> If set theorists were not motivated by a maxim of this sort [UNIFY], there would be no pressure to settle CH ... or to choose between alternative new axiom candidates; it would be enough to consider a multitude of alternative set theories. [Maddy, 1997, 209]

However, as forcing has become a central tool in contemporary set theory, some scholars regard it as a challenge to the universe view, precisely based on arguments from set-theoretic practice.

2.3 Forcing in the Philosophy of Set Theory

A major transformation in set-theoretic practice was brought about by the introduction of the forcing technique. Developed by Cohen in 1963 to complete the independence proof of CH, forcing has since become one of the central achievements of modern set theory. Cohen's method provided set theorists with a powerful and flexible tool for constructing models of set theory, and it soon proved exceptionally effective for establishing consistency results.

The most common approach to forcing is the countable transitive model approach, which can be summarised as follows. To determine whether a statement A is consistent with ZFC—that is, whether it *can* be true—one typically aims to construct a model in which A holds. A could be, for instance, $\neg$CH. Given a countable transitive model M that satisfies the axioms of ZFC, the forcing method begins with the definition of a partial order P in M, consisting of small fragments of the set one wishes to obtain. To prove that $\neg$CH is consistent, one aims to construct a bijective function between the continuum and a cardinal larger than $\aleph_1$, most simply $\aleph_2$, and the partial order P consists of fragments of such a function. Since M is countable, the partial order P, as an element of M, is also countable from an external perspective; however, M itself cannot prove that P is countable. This reflects the counter-intuitive existence of countable models of set theory, guaranteed by the Löwenheim-Skolem theorem, and highlights the crucial shift in perspective between reasoning inside M and outside M. From the external viewpoint, one can define a so-called generic object G, built from the fragments in P, and prove its existence. Yet, since M cannot establish the countability of P, the object G does not exist *in* M, but only *outside* M. Using G, one constructs an extended model $M[G]$, which satisfies the axioms of ZFC and contains G, and, if P has been chosen appropriately, also contains the desired set, such as a bijection between the continuum and $\aleph_2$. In this way, forcing combines meta-theoretic and object-theoretic aspects: while the overall method operates at the meta level by constructing models from an external standpoint, its execution relies on detailed combinatorial work within the object theory, namely the design and analysis of partial orders P that ensure the truth of A while preserving sufficient structural similarity to the ground model. For reasons of space, this chapter cannot introduce the mathematical details of the forcing method in more depth;

the interested reader is referred to the standard textbooks [Kunen, 1980] and [Jech, 2003, Ch. 14–15], or to the introductory articles [Easwaran, 2007] and [Chow, 2008].[8]

The method can also be iterated transfinitely, allowing forcing stages to be performed successively through infinitely many steps to obtain the desired properties in the final model. Over the decades, numerous forcing notions have been developed, each tailored to achieve specific goals: Cohen forcing, Easton forcing, Laver forcing, Sacks forcing, proper forcing, and stationary set preserving forcing, among many others.[9]

Forcing has also inspired a range of forcing axioms—such as Martin's Axiom (MA), PFA, and Martin's Maximum (MM)—as well as more specialized principles such as (*).[10] Large cardinal axioms often play a crucial role in forcing arguments, as strong large cardinal hypotheses can enable constructions that would otherwise be impossible. Moreover, forcing is not confined to independence proofs: through techniques involving absoluteness and related ideas, it can also yield theorems of ZFC itself (a prominent example is Silver's theorem [Silver, 1975]). Because of its wide applicability, forcing is now employed throughout virtually all areas of set theory.

Forcing is not only mathematically significant but also philosophically rich. The proliferation of forcing extensions has prompted extensive reflection on the ontology of set theory, most notably in the work of Hamkins, who advances a multiverse view according to which the abundance of forcing-generated models reveals the true structure of the set-theoretic realm. Before going into Hamkins' ideas in more detail, let us review a few other philosophical discussions involving forcing.

Forcing is a powerful technique for constructing new models of set theory from a given one. At first glance, this ability to generate distinct models seems to pose a direct challenge to the universe view. Consider the set-theoretic universe, denoted by V, which is supposed to contain all sets. By applying forcing, we can construct sets that lie outside a given model, suggesting that the model did not, in fact, include all sets. One might therefore be tempted to conclude that the very notion of V is incoherent.

A natural first response is to observe that V is a proper class, whereas forcing is carried out only over set-sized models. Yet the challenge does not disappear so easily, since such set models are often regarded as small copies of the set-theoretic universe itself.

Nevertheless, as Neil Barton [2020] points out, the universist has coherent ways of interpreting forcing without falling into contradiction. Forcing is applied only to countable transitive models of set theory, all of which lie within V, as do their generic extensions. Moreover, forcing is not merely compatible with universism—it is instrumentally valuable to it. It can be used to prove theorems (for instance, via absoluteness arguments) and to motivate or formulate new axioms. Indeed, several forcing axioms have been proposed, and some universist-minded set theorists regard these as promising candidates for extending ZFC.

Within the philosophical literature, one line of discussion concerns different conceptualisations of forcing, evaluated according to virtues such as simplicity, naturalness,

[8]For readers of German, I also wrote an introduction to independence and forcing [Kant, 2025d].

[9][Jech, 2003, Ch. 16].

[10][Jech, 2003, Ch. 31, 36].

and the plausibility of their underlying philosophical assumptions—often in connection with potentialism.[11] Most of these philosophical treatments, however, are not directly reflected in the mathematical practice of set theory. They involve considerations primarily relevant to philosophical interpretation rather than to the way set theorists actually employ forcing. Consequently, they cannot be substantially informed by set-theoretic practice. Despite their intrinsic philosophical interest, I set these approaches aside in the present discussion.

A second strand in the literature is more closely intertwined with the mathematical use of forcing. It concerns the development of various multiverse conceptions [Antos et al., 2015]. Forcing can be iterated to generate not merely a single extension but a rich array of models, which may be assembled into larger multiverse frameworks. It is therefore a central tool in constructing multiverse structures. Sy-David Friedman introduced the hyperuniverse—the collection of all "nice" countable transitive models of set theory—as a tool for searching for new axioms [Arrigoni and Friedman, 2013]. W. Hugh Woodin [2011] developed the generic multiverse, the collection of all generic extensions of a given model, with the aim of uncovering new set-theoretic truths. John Steel [2014] proposed an axiomatic multiverse theory incorporating universes with large cardinals. Interestingly, this multiverse includes an identifiable core model and is therefore viewed as compatible with a universe conception of sets.[12] Jouko Väänänen [2014] has formulated yet another multiverse framework aimed at conceptualizing the notion of absolutely undecidable sentences, while leaving open the possibility that none exist. Surveying these approaches, Giorgio Venturi [2016, 2020] argues that studying multiverse structures from multiple perspectives helps to develop more robust intuitions about generic sets. This, he notes, coheres with one of the fundamental aims of set theory: to understand the notion of arbitrary sets.

These diverse interpretations illustrate how forcing has become not only a central mathematical tool but also a testing ground for competing philosophical conceptions of set theory.

These multiverse frameworks were developed for quite different purposes, and this diversity should be kept in mind. Even set theorists such as Friedman and Woodin—who sought new axioms beyond ZFC—engaged extensively with multiverse structures, despite advocating positions that stand in sharp contrast to Hamkins' pluralism.

2.4 Hamkins' Multiverse View and Practice-Based Pluralism

The most prominent philosophical alternative to the universe view is presented by Hamkins [2012]. For him, the actual research practices of set theorists—especially their use of forcing and model-building—motivate very different philosophical conclusions.

Briefly summarised, Hamkins argues that the multiverse view provides a coherent account of the day-to-day experience of set-theoretic practitioners. His argument is practice-based, drawing directly on how set theorists work. In his view, the ubiquity of

[11]Without attempting to be exhaustive, see, for example, Roberts [2025], Barton [2025], Brauer [2025], and de Ceglie [2021].

[12]I'll say more about this in Sec. 2.5.

forcing and the continual construction and comparison of models of set theory strongly support the Platonic existence of a vast set-theoretic multiverse. At the same time, he emphasises that this need not undermine the foundational role of set theory. Let us examine his position in more detail.

Hamkins' multiverse view denies the existence of a unique set-theoretic universe. Instead, it posits a multiverse: a vast collection of set-theoretic universes. He stresses that this multiverse is extremely liberal:

> There seems to be no reason to restrict inclusion [in the multiverse] only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP, and so on. [Hamkins, 2012, 436]

One of his most controversial multiverse axioms is *Well-Foundedness Mirage*, which states that every set-theoretic universe is ill-founded from the perspective of another universe. Hamkins wants to emphasise with this axiom that the property of being well-founded depends on the background theory [Hamkins, 2012, 439]. A prime example for this phenomenon are non-standard models of arithmetic. Internally, they are well-founded, but from an external perspective (i.e. from the perspective of a model of the meta-theory), non-standard models are ill-founded. Therefore, this principle implies the inclusion of many counter-intuitive models within Hamkins' multiverse.

Hamkins' main argument for the multiverse is its explanatory power with respect to practice. Set theorists routinely work with a wide variety of models, and Hamkins insists that the multiverse view captures what practitioners *actually do*, rather than what they say they are doing:

> regardless of what set theorists might assert, their mathematical behaviour demonstrates that they accept the existence of diverse set-theoretic worlds. [Hamkins, 2015, 136]

From this perspective, the multiverse view is presented as a serious theoretical framework intended to explain the research activities of set theorists. This close connection to set-theoretic practice is the main reason it directly conflicts with Maddy's view.[13]

In contrast to Maddy's principle UNIFY, which relies on straightforward reasoning about set theory's foundational role, a pluralism of Hamkins' kind appears difficult to reconcile with the idea that set theory provides a unified foundation for mathematics, since questions beyond ZFC may lack determinate answers. Hamkins, however, responds:

> When a mathematical issue is revealed to have set-theoretic dependence ... then the multiverse response is a careful explanation that the mathematical fact of the matter depends on which [universe] is used, and this is almost always a very interesting situation, in which one may weigh the desirability of various set-theoretic hypotheses with their mathematical consequences. [Hamkins, 2012, 419]

[13] I have examined in greater detail the relationship between Hamkins' multiverse view and set-theoretic practice in [Kant, 2025b]. The present chapter broadens that discussion by situating it within the more general context of practice-based arguments.

Thus, Hamkins maintains that set theory retains its foundational role, albeit in a pluralistic form.[14]

2.5 Comparing Maddy and Hamkins: A Methodological Puzzle

Because Hamkins' multiverse view has been widely discussed in the philosophy of set theory, Maddy explicitly addresses it. To assess the issue systematically, she considers the goals successfully achieved by a universe theory of sets and asks whether a multiverse theory could accomplish them equally well—or perhaps serve different purposes instead. According to Maddy, a universe theory of sets fulfils several foundational roles [Maddy, 2017, 2026]:

∗ Meta-mathematical Corral: providing a syntactic setting in which all mathematical reasoning can be carried out.

∗ Elucidation: offering precise set-theoretic definitions of mathematical objects.

∗ Risk Assessment: measuring the consistency strength of new theories.

∗ Shared Standard of Proof: sustaining a common standard for what counts as a proof.

∗ Generous Arena: supplying a rich domain encompassing all kinds of mathematical objects.

Because Hamkins' multiverse lacks a precise axiomatization, Maddy argues that it cannot fulfil these roles to the same extent as a universe theory of sets. She therefore concludes that, while the multiverse may serve a valuable heuristic purpose—for instance, in areas such as set-theoretic geology or in Steel's multiverse thinking—it cannot replace traditional set-theoretic foundations:

> In sum, then, it seems to me that the familiar set-theoretic foundations, rough and ready as they are, remain the best tool we have for the various important foundational jobs we want done. [Maddy, 2017, 317]

Recent work has led her to adopt a slightly modified position in the universe–multiverse debate. In Maddy [2024], she develops a general account of multiversism and distinguishes several explicit versions of both multiversism and universism:

∗ Metaphysical universism / multiversism

[14] There is another argument linking set theory's foundational role with set-theoretic pluralism. In current mathematical practice, axioms beyond ZFC are rarely applied outside set theory itself, and even less often required to resolve extra-set-theoretical problems. A prominent but rare exception is Ilijas Farah's work on the Calkin algebra, which employs the open colouring axiom [Farah, 2011, 2019]. This situation suggests that new axioms cannot be justified in the same way as, for example, the Axiom of Choice, which was largely accepted due to its central applications across mathematics. If this situation persists, only ZFC can be justified by appealing to set theory's foundational role: additional axioms simply do not appear necessary for founding mathematics.

* Axiomatic universism / multiversism
* Heuristic universism / multiversism

She rejects both metaphysical universism and multiversism, endorses both heuristic universism and multiversism, and further endorses axiomatic universism while rejecting axiomatic multiversism. However, the latter stance appears largely to reflect a pragmatic preference for a universe-based axiomatic framework, since Maddy is otherwise highly sympathetic to Steel's multiverse theory MV [Steel, 2014]. This marks the main shift from her earlier views [Maddy, 2024, 2026]. In Steel's theory, there are two types of variables, ranging over sets and over universes. Each universe contains all large cardinals, and the universes satisfy amalgamation (i.e. for any two universes, there exists a larger universe extending both). Although this multiverse is less expansive than Hamkins' multiverse, it has a more structured organisation. In particular, it includes a distinguished core model C, which is contained in all other universes. Through C, one can access all other universes, since they arise as forcing extensions of C. Conversely, any universe can be connected to any other by passing through C. Within this framework, the principle Unify is interpreted as the unification of all natural extensions of ZFC together with their interrelations. If one identifies the set-theoretic universe V with the core model C, then these various models can be systematically studied. Steel and Maddy therefore regard $V = C$ as an attractive axiom candidate. While this does not resolve the continuum problem, it may be supplemented by Woodin's axiom $V = \text{Ultimate-}L$.[15]

According to Maddy's terminology, an axiomatic multiversist would favour $\text{MV} + \text{LCA} + V = C$, whereas an axiomatic universist would favour $\text{ZFC} + \text{LCA} + V = C$; however, these theories are, in effect, interchangeable. This helps explain why the rejection of axiomatic multiversism is primarily pragmatic: working with an extension of ZFC is standard practice and therefore methodologically more straightforward for set theorists [Maddy, 2026, 20–23].

Hamkins' multiverse view is not a form of axiomatic multiversism. In Maddy's terminology, Hamkins endorses metaphysical multiversism. Maddy therefore argues that his position is not directly opposed to her own, namely axiomatic universism. This raises the question: where, exactly, does the conflict between their views lie? A closer look at their respective positions on the continuum problem helps to clarify the disagreement. Maddy writes:

> I believe that one of the most pressing questions in the contemporary foundations of set theory is how to extend ZFC (or ZFC+LCs) in mathematically defensible ways so as to settle CH (and other independent questions) and to produce a more fruitful theory. [Maddy, 2024, 505]

For her, resolving the continuum problem is not only a meaningful goal, but apparently the most pressing one in contemporary set theory. For Hamkins, by contrast, the

[15] The idea behind this axiom candidate is that the set-theoretic universe is a canonical inner model containing all large cardinals. However, the construction of such an inner model remains an open mathematical problem. A philosophical analysis of Woodin's shift toward this axiom candidate is provided by Rittberg [2015].

continuum problem is already settled:

> On the multiverse view, ... the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties. [Hamkins, 2012, 429]

In addition to this direct disagreement over the status of the continuum problem, Maddy's axiomatic universism is not readily compatible with Hamkins' metaphysical multiversism—or at least, no well-developed position combines the two in a convincing way. These considerations bring out some of the conceptual subtleties underlying the debate.[16] In what follows, I use "the universe view defended by Maddy" to refer to her axiomatic universism together with her stance on the continuum problem, and "Hamkins' multiverse view" to refer to his metaphysical multiversism together with his stance on the continuum problem.

Given the opposing positions of Maddy and Hamkins—an axiomatic universe view versus a metaphysical multiverse view—and the fact that both appeal to set-theoretic practice to justify their stance, a methodological puzzle about how practice is selected and weighted arises: how can they draw such conflicting conclusions from the same empirical facts about set-theoretic practice? Where, precisely, does the divergence come from?

3 The Scope Guideline: Interpreting Set-Theoretic Practice

To begin addressing this puzzle, it is helpful to consider a methodological issue that may account for at least part of the divergence. The explanation is relatively straightforward: Maddy and Hamkins may be referring to different aspects of set-theoretic practice—different communities of set theorists or distinct methodological tendencies—and thus arrive at contrasting conclusions. This raises a broader question: how can a philosophical account of set theory do justice to the full diversity of the set-theoretic community?

Rittberg [2016] was the first to argue that Maddy overlooks significant portions of set-theoretic practice in her analysis and therefore defends a universe view on the basis of overly simplified empirical observations. After all, Hamkins is himself a practising set theorist, yet he does not pursue the kinds of goals that Maddy attributes to the "universists" she takes as representative for extracting the proper methodology of set theory. According to Rittberg, the set-theoretic goals Maddy identifies are therefore not grounded in genuine consensus. Recall that in step 1 of Maddy's procedure, consensus is essential for extracting set-theoretic goals from practice.

Indeed, Maddy explicitly specifies in her earlier work:

> I will concentrate on the views of the Cabal seminar, whose work centers on determinacy and large cardinal assumptions. Along the way, especially in the

[16] I am grateful to Penelope Maddy for encouraging me to consider these details.

> early sections, the views of philosophers and set theorists outside the group, and even opposed to it, will be mentioned, but my ultimate goal is a portrait of the general approach that guides the Cabal's work. [Maddy, 1988a, 482]

The Cabal group was an influential research collective active in California during the 1970s and 1980s, dedicated to the study of large cardinal axioms and determinacy principles. Its members included Matthew Foreman, Steve Jackson, Alexander Kechris, Donald Martin, Yiannis Moschovakis, Robert Solovay, Steel, and Woodin. It is methodologically problematic, however, to extract consensus-based set-theoretic goals while drawing exclusively from such a restricted circle of practitioners.

The methodological lesson is clear:

> **Scope guideline:** A practice-based account of set theory should respect—and be able to explain—the full diversity of set-theoretic practices, or else explicitly restrict its scope to a specific context.

Like Maddy's, Hamkins' practice-based account also fails to comply with this scope guideline. Although the practice of forcing is indeed widespread, his conclusion that there is no unique set-theoretic universe stands in sharp tension with the research aims of absolutist practitioners who seek to extend ZFC by discovering new true axioms. A parallel problem thus arises for both Maddy and Hamkins: Maddy neglects, and therefore cannot explain, pluralist practices; Hamkins neglects, and therefore cannot explain, absolutist practices. Yet both strands are integral to contemporary set-theoretic practice [Kant, 2025a].

Failing to satisfy the scope guideline does not show that a practice-based account is false, but that it overgeneralizes from a restricted segment of practice and therefore lacks explanatory adequacy as a general philosophy of set theory.

However simple the scope guideline may appear, its implementation is far from easy. A general account of set theory is, of course, philosophically more attractive than one narrowly restricted—for example, to the practices of the Cabal group. How, then, can we comply with the scope guideline?

First, we can respect the full diversity of set-theoretic practices by integrating multiple forms of practice into a single philosophical account. The empirical observations on which such an account rests should be broad in scope and deliberately inclusive of diverse practices. At the same time, these observations must be weighed carefully: widespread practices should carry greater evidential weight than isolated or idiosyncratic ones.

Second, once an account has been developed on the basis of certain practices, we must seek to *explain* divergent practices when they arise. Only then can we hope to make philosophical claims about set theory *in general*. For example, it is entirely conceivable to explain pluralist practices within Maddy's framework. Pluralist set theorists appear to pursue at least some goals that differ from those of absolutists—such as exploring possibilities: What can be true? What cannot? Pluralists reject the UNIFY principle and instead place their trust in the exploratory power of ZFC. But if one exploits ZFC to investigate the landscape of possible truths, then constructing a wide array of forcing models and developing specialized forcing techniques is an effective means to that end.

This is only a sketch rather than a full argument, but it illustrates the kind of interpretive work required to satisfy the scope guideline.

4 Case Study: Forcing and the Diversity of Practice

4.1 Motivation: Is Forcing Philosophically Neutral?

Let us now apply the scope guideline to a concrete philosophical question concerning the practice of forcing. Forcing is widely employed across the set-theoretic community—including by absolutist practitioners. Woodin, for instance, studied the generic multiverse in his early search for new axioms. Steel, another absolutist-minded set theorist, developed an axiomatic multiverse theory that allows discourse about multiple universes. Moreover, several set theorists, such as Joan Bagaria, Menachem Magidor, and Matteo Viale, have argued that ZFC should be extended by forcing axioms.[17]

If forcing is thus compatible with both a universe view and a pluralist view, might the practice of forcing be philosophically neutral?

In what follows, philosophical neutrality should not be understood as a strong claim that forcing lacks any philosophical implications whatsoever. Rather, the question is whether the use of forcing, as a general methodological practice, determines a commitment to a particular philosophical position, or whether it remains compatible with distinct interpretations depending on the research aims it serves.

4.2 Empirical Findings: Absolutist and Pluralist Practitioners

The scope guideline advises us to examine different practices of forcing. Accordingly, this section explores perspectives on forcing from both absolutist and pluralist standpoints. The analysis draws on a qualitative interview study with 28 professional set theorists.[18] The sample is diverse in research specialization,[19] academic generation,[20] gender,[21] and geographic location.[22] The study's methodology and detailed analyses are presented in

[17] Bagaria [2000] argues for the adoption of bounded forcing axioms. Magidor [2022] argues for adopting forcing axioms—specifically MM^{++}—conditional on a conjecture since proven true [Asperó and Schindler, 2021]. Viale [2016] offers a philosophical justification for forcing axioms. Comprehensive treatments by Justin Moore [2017] and Stevo Todorčević [1997] can also be interpreted as philosophical arguments in favour of forcing axioms, though without the explicit call to adopt them as true axioms found in the previously mentioned works.

[18] According to the "list of homepages of set theorists", managed by set theorist Jean A. Larson, the sample comprises of 8.3% of all set theorists (28 of 338). All participants held or had held permanent academic positions as professors of mathematics with a research focus on set theory.

[19] Participants' research areas included combinatorics, descriptive set theory, inner model theory, forcing axioms, large cardinals and forcing, forcing, set-theoretic and general topology, and cardinal characteristics. Each listed area was mentioned by at least four participants.

[20] The sample represents multiple generations: six participants earned their PhD before 1980, four between 1980 and 1989, nine between 1990 and 1999, and nine after 1999.

[21] Twenty-four participants were male, roughly reflecting the overall gender balance within the set-theoretic community.

[22] Fifteen participants were affiliated with European universities and eleven with universities in the United States, the two major centres of set-theoretic research.

[Kant, 2025a,b,c]; a structured summary of empirical findings and methods is available in an online repository (REFERENCE forthcoming). Although the study is not statistically representative, the breadth of the sample is sufficient to test philosophical accounts that aim to characterize set-theoretic practice in general rather than the views of a narrowly defined sub-community.

A central topic of the study concerned set theorists' views on forcing. Participants were asked several targeted questions about it,[23] revealing several distinct philosophical orientations within the community. Some participants identified as absolutists, believing that ZFC should be extended by new, true axioms. Others expressed pluralist views, maintaining that ZFC is sufficient and that independence is an intrinsic feature of set theory. A few participants did not align neatly with either position. Some of them held strong but mixed views, for instance having favourite axiom candidates, but pointing out that the whole mathematical community would have to decide on new axioms, and not only set theorists. For others, the philosophical issues were rather distant from their every-day work experience, because their specific research area was concerned with matters that can be settled in ZFC. All participants, however, were familiar with forcing and could discuss it competently. To illustrate the sharpest contrasts, I present the perspectives of the absolutist and pluralist practitioners in turn.

To protect the anonymity of the participants, I will quote participants in the study anonymously in what follows and in such a way that different quotations cannot be attributed to the same person. In selecting the quotations, I took care to cite different participants.

Absolutist practitioners. Most absolutist participants expressed belief in the set-theoretic universe V and the overarching goal of identifying the "right" new axioms. They view ZFC as insufficient for settling important problems, noting partial progress toward determining axioms for $H(\omega_1)$ and an ongoing effort to settle those for $H(\omega_2)$.[24] The findings indicate that absolutist practitioners typically accept large cardinal axioms (at least up to Woodin cardinals) and PD—the same justificatory reasons for which align with those identified in [Maddy, 2011, 47–51].

[23] The key questions included:

* What do you say about the view that forcing was considered very unnatural when it was introduced?
* What do you say about the view that forcing is today a natural part of set theory?
* Can you describe the difference between using a specific forcing notion and applying a forcing axiom?
* Do you think that one can distinguish in set theory between an object level, where one talks about sets, and a meta level, where one talks about models of set theory—or do you think that such a distinction is inappropriate for set theory?
* Some set theorists are looking for new axioms to extend ZFC; others think that such aims are pointless. Do you have an opinion on that?

Additional remarks on forcing were often made when participants described their research areas or discussed related topics. For the complete questionnaire, see [Kant, 2025a, Sec. 3.1.2].

[24] See [Woodin, 2001] for a description of this stepwise approach, which participants also referenced.

Addressing Hamkins' multiverse view directly, many absolutist participants explicitly rejected it:

> I don't like the multiverse view. I don't think it describes really what we do.

(For a more detailed account of such absolutist views, see [Kant, 2025a, Sec. 4.2.2].)

Pluralist practitioners. Pluralist participants typically maintained that ZFC will not be extended by new axioms:

> If I had to guess, I don't really expect any axioms beyond ZFC to be adopted and have the same status in the mathematical community that the axioms of ZFC do.

They are content with ZFC and its intrinsic independence phenomena:

> I'm perfectly happy with ZFC and a pluralism of universes.

For them, working with diverse axioms and models is intellectually valuable:

> I don't feel the need to choose between axiom systems. They're all out there, and you can study all of them—that's all worthy of study.

Some pluralists described an algebraic attitude toward set-theoretic models, comparing them to algebraic structures like groups or fields—objects where it is unsurprising that certain properties hold in one structure but not in another. Extending ZFC by new axioms, from this perspective, offers little benefit, since it would restrict the variety of available models. A few participants highlighted their appreciation for the flexibility of working across models:

> That's something I find fascinating in set theory—that your world is not fixed. You can manipulate things, and not everything is determined as true or false. You can play with further truths, have some room to manoeuvre—then it becomes less boring.

(For further examples, see [Kant, 2025a, Sec. 4.2.3].)

Assessing the philosophical neutrality of forcing. Having identified these two subgroups, we can now assess whether the practice of forcing is philosophically neutral. The following two observations emerge from the data:

Observation 1: Absolutist practitioners value forcing and find it compatible with their world-view, but they place little value on the construction of technologically highly complex forcing models, such as those used in research on cardinal characteristics.

Observation 2: There may be a correlation between a pluralist world-view and the practice of constructing technologically highly complex forcing models—for example, in the study of cardinal characteristics.

Evidence for Observation 1. Among the eleven absolutist participants, views on the philosophical implications of forcing varied. Four were proponents of forcing axioms, arguing that adopting such axioms acknowledges forcing as a logical and mathematical fact. Another argument was that one can discuss forcing extensions within the set-theoretic universe V; hence, it is simpler to assume that the objects of those extensions already exist in V.

Others, however, questioned the philosophical significance of forcing for discovering new axioms:

> Forcing has found its way into all parts of set theory. Even inner model theorists use it when they're building models. ... Everybody needs to use forcing in some way. ... But whether the forcing models themselves tell us something about V, that's another philosophical question altogether.

Such participants maintained that forcing "is not about V." Some compared forcing constructions to linguistic artefacts, akin to non-standard integers.

Furthermore, absolutist practitioners tended to regard separation results—for instance, consistency results in the area of cardinal characteristics—with limited enthusiasm:

> For what concerns consistency results, I find it mostly a game now, because the most interesting ones have been proved. In my perspective, it's better to find new arguments to choose among the many possibilities which is the right one.

Another participant commented:

> I don't think this [consistency result about cardinal invariants] tells us anything about V or the actual real reals. It only shows what we can't prove in ZF about these reals.

Evidence for Observation 2. Table 1 presents data correlating philosophical stance with research area.

Table 1: Philosophical views according to research areas

Total (interviewees)	[absolutist] 11	[pluralist] 11	neither 6
Combinatorics	5	5	3
Descriptive set theory	5	1	5
Inner model theory	4	3	1
Forcing axioms	4	3	1
Large cardinals and forcing	3	4	1
Forcing	2	6	0
Set-theoretic/general topology	2	3	0
Cardinal characteristics	0	4	0

While most research areas are represented across both philosophical orientations, the data suggest a possible correlation between pluralism and work on cardinal characteristics.[25] Though more representative data would be needed to confirm this, such a correlation would be conceptually plausible. Research on cardinal characteristics involves studying uncountable cardinal invariants of the continuum. If CH holds, all these invariants equal $\aleph_1$. Yet the central research question is whether distinct invariants can differ in size. Addressing this requires constructing technologically highly complex forcing models in which CH fails and the continuum takes a larger value. For example, in recent work on Cichoń's Maximum, ten distinct cardinals between $\aleph_0$ and $2^{\aleph_0}$ are realized, implying that the continuum has at least size $\aleph_{10}$ in the constructed extension.[26] Such work presupposes—and thrives on—the flexibility afforded by a paradigmatic pluralist stance toward the continuum.

4.3 Interpreting the Results

Forcing, as we have seen, is integral to both absolutist and pluralist practices in set theory. While absolutist practitioners acknowledge its utility, they diverge in their interpretations: some advocate the adoption of forcing axioms, whereas others question the philosophical relevance of forcing models for the discovery of new axioms, and view the proliferation of specific consistency results with scepticism. By contrast, pluralist practitioners—especially those working on cardinal characteristics—embrace forcing as an exploratory tool that is essential to their research aims.

The practice of forcing can be coherently interpreted within both Maddy's and Hamkins' frameworks. From a Maddy-style rational reconstruction, different research communities pursue distinct goals, and thus embrace different means to achieve their goals. Set theorists working on cardinal characteristics aim to explore potential truths in fine detail, employing highly technical forcing constructions as an effective means toward this pluralist goal. Others, by contrast, seek a unified and coherent theory of sets that settles open questions and remains stable under forcing; for some of them, forcing axioms function as the corresponding means to this absolutist end.

From the perspective of Hamkins' practice-based pluralism, practitioners investigate a range of available set-theoretic universes. Some aim to identify the best among these universes, guided by shared evaluative criteria such as naturalness, fruitfulness, or elegance. While there is broad agreement that universes differ with respect to such virtues, a particular subgroup of practitioners seeks to maximise these virtues within a single, distinguished universe.

This analysis shows that the everyday practice of forcing across the set-theoretic community is, in principle, compatible with both Maddy's and Hamkins' philosophical positions, but only if certain stronger claims are relaxed. In Maddy's case, the pursuit of a unified theory of sets is not a universal research goal. In Hamkins' case, the practice of forcing does not support Platonism about set-theoretic universes. Rather, it supports

[25] See [Kant, 2025a, Sec. 4.2.4] for a full discussion of this table. Participants could indicate multiple research areas.

[26] [Goldstern et al., 2019, 2021].

the more modest claim that set-theoretic models, alongside sets, are fundamental entities of set theory [Antos, 2022]. In other words, the philosophical assumptions that generate the conflict between Maddy and Hamkins are not grounded in set-theoretic practice.

Independently of these frameworks, the data show that forcing is widely valued across the set-theoretic community, albeit for different reasons. Its enduring significance lies precisely in its versatility: forcing functions as a flexible and powerful tool that can be employed in pursuit of diverse mathematical and philosophical aims. Overall, the practice of forcing is compatible with multiple philosophical interpretations. A notable exception is research on cardinal characteristics, where the methodology depends so deeply on the construction and comparison of distinct models that a pluralist interpretation seems most appropriate.

5 Conclusion

The analysis presented in this chapter has traced how one of set theory's most central techniques—forcing—interacts with philosophical interpretation. From the universe view defended by Maddy to Hamkins' multiverse pluralism, both accounts appeal to set-theoretic practice but extract sharply different lessons. Examining their divergence through the scope guideline reveals that neither framework, taken alone, captures the full diversity of contemporary practice. A satisfactory philosophical account must therefore explain not only the existence of multiple research aims but also how they coexist within a single mathematical discipline.

The case study of forcing shows that philosophical underdetermination by practice is only partial. Forcing is universally valued because of its exceptional methodological flexibility: it can be used to explore the landscape of possible models, to articulate new axioms, or to investigate the stability of existing ones. Yet some applications—particularly in the study of cardinal characteristics—depend so fundamentally on the plurality of models that they presuppose a pluralist position. In this limited sense, forcing embodies both the unity and the diversity of set-theoretic research.

Taken together, these findings underscore a broader moral for practice-based philosophy of mathematics. Set-theoretic practice is not monolithic: it encompasses distinct yet interconnected methodological traditions, each with its own epistemic ideals and philosophical resonances. Rather than seeking a single interpretation, philosophers should aim to articulate frameworks capable of doing justice to this internal diversity. Seen through the lens of the scope guideline, this diversity is not a defect to be eliminated, but a feature that any credible practice-based philosophy of set theory must be able to explain. The practice of forcing thus offers not only deep insights into set theory itself but also a paradigm for how philosophical analysis can remain responsive to the evolving realities of mathematical work.

Looking at mathematical practice more generally, the present investigation raises the question of whether the strong diversity of philosophical views observed in the set-theoretic community generalises to the mathematical community as a whole. Do most mathematicians also hold comparably strong absolutist or pluralist convictions?[27] In the absence of systematic data, I offer a few tentative reflections. A key difference between set-theoretic research and most other areas of mathematics lies in the impact of the independence phenomenon. Set theorists routinely work with different axiomatic systems, since much of their subject matter varies with the choice of such systems. In most other areas of mathematics, this is *not* the case. One might therefore suspect that the presence of the independence phenomenon requires set theorists to adopt more determinate philosophical positions than mathematicians in other fields.

Indeed, while I would expect that mathematicians outside set theory do not typically hold strong absolutist or pluralist views comparable to those found among set theorists, it is nevertheless plausible that they exhibit considerable diversity in their individual philosophical beliefs. These views are likely to be more closely tied to familiar philosophical positions, such as Platonism or formalism. At the same time, there may be also correlations between specific areas of research and particular philosophical convictions. One example for this is the constructive orientation often found among mathematicians working in homotopy type theory.

Overall, this suggests that the extent and character of philosophical diversity in mathematics are shaped by the specific features of each domain, and that the scope guideline provides a sound methodological framework for practice-based philosophy more generally.

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[27] Thanks to Alexander Paseau for encouraging me to address this question.

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